



Cauchy's integral formulation of Bateman equations for successive radioactive transformations

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Manuscript details:

Available online on <http://www.ijlsci.in>

ISSN: 2320-964X (Online)

ISSN: 2320-7817 (Print)

Cite this article as:

Meghwal Ram Swaroop (2025) Cauchy's integral formulation of Bateman equations for successive radioactive transformations, *Int. J. of. Life Sciences*, Special Issue, A21, 187-193.

Article published in Special Issue of "International Conference on Futuristic Approach for Viksit Bharat (ICFAVB-2047) Organized by Government College, Dholpur (Raj.) Dated 23 and 24 Dec. 2024.



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ABSTRACT

Cauchy's formula for integration of a function of complex variable is applied to radioactive disintegration. Harry Bateman's formula for successive radioactive decays converted into an elegant form. Mathematical singularities suggested peculiar instabilities in the physical system. The singular points in λ - domain, are the localities of radioactive decay of the physical system.

Keywords: radioactive decay, Harry Bateman's formula, Cauchy's formula, λ -domain.

1. INTRODUCTION

The theoretical understanding about the system when no exact knowledge regarding the forces binding the constituent particles is often derived from the spontaneous response of the system. In this context the study of radioactivity is inevitable in nuclear science. The radioactive disintegration is quantum mechanical in the sense that we will not be able to predict when exactly a particular nuclide will disintegrate in a given time in an ensemble. The phenomenon is of statistical in nature and we shall predict the number of nuclides that will probably decay in an ensemble in selected time duration.

2. FUNDAMENTAL LAW OF RADIOACTIVITY

E. Rutherford and F. Soddy quantitatively described the experimentally observed radioactive phenomenon as an exponential decay [1, 2]. In successive radioactive transformations, the parent nuclide decays into a daughter nuclide, which decays in turn and so on and the process goes on till the stable end product yields. The equations of exponential decay form a system and mathematical tool is required to handle the problem. Harry Bateman suggested the route to overcome the tedious procedure of finding solution [3]. Bateman solution exhibits the decay to be exponential. Towards the path of generalization many researchers contributed appreciably [4-7]. But the conciseness is still missing in the literature.

I made my efforts to achieve it and I found Cauchy's formula of contour integration to suffice the path.

The general equation for decay process can be regarded as :

$$\frac{dN}{dt} = -\lambda(N, t)N^n$$

The decay parameter λ , in general, might depends upon the nature of the substance, number of decaying entities and the time.

The number 'n', may be natural, whole, real or complex.

To simplify the problem, we assume

$$\lambda(N, t) = \lambda(0, 0) = \text{constant}$$

The decay rate must also depend upon the number of decaying entities. In general, we may have some power for N.

$$\frac{dN}{dt} = -\lambda N^n$$

$$n \in \mathbb{C}$$

$$n = 0, 1, 2, \dots \in \mathbb{W}$$

$$n = 1 \Leftrightarrow \text{exponential decay}$$

With these conditions, decay is said to be exponential. This is Rutherford-Soddy formulation of decay process.

3. BATEMAN SOLUTION FOR SUCCESSIVE DECAY AND USE OF CAUCHY FORMULA

The parent system is described by the population N_1 and daughters are designated by populations N_2, N_3, N_4 (in our case for generalization) and the stable end product formed is denoted by population N_5 . The radioactive decay is well theorized by disintegration constants $\lambda_1, \lambda_2, \lambda_3, \lambda_4$. In this way we get the following system of equations.

$$\left. \begin{aligned} \frac{dN_1}{dt} &= -\lambda_1 N_1 \\ \frac{dN_2}{dt} &= -\lambda_2 N_2 + \lambda_1 N_1 \\ \frac{dN_3}{dt} &= -\lambda_3 N_3 + \lambda_2 N_2 \\ \frac{dN_4}{dt} &= -\lambda_4 N_4 + \lambda_3 N_3 \\ \frac{dN_5}{dt} &= \lambda_4 N_4 \end{aligned} \right\}$$

...(1)

Following Bateman the solution is as obtained below:

$$e^{\lambda_1 t} \frac{dN_1}{dt} + \lambda_1 e^{\lambda_1 t} N_1 = 0 \cdot e^{\lambda_1 t} \Rightarrow \frac{d(N_1 e^{\lambda_1 t})}{dt} = 0 \Rightarrow N_1 = N_1^{(0)} e^{-\lambda_1 t} = C_1^{(1)} e^{-\lambda_1 t} \quad \dots(2)$$

$$e^{\lambda_2 t} \frac{dN_2}{dt} + \lambda_2 e^{\lambda_2 t} N_2 = \lambda_1 N_1^{(0)} e^{\lambda_2 t} e^{-\lambda_1 t} \Rightarrow \frac{d(N_2 e^{\lambda_2 t})}{dt} = \lambda_1 N_1^{(0)} e^{(\lambda_2 - \lambda_1)t}$$

$$N_2 e^{\lambda_2 t} = \lambda_1 N_1^{(0)} \frac{e^{(\lambda_2 - \lambda_1)t}}{\lambda_2 - \lambda_1} + \text{const.} \Rightarrow N_2 = \lambda_1 N_1^{(0)} \frac{e^{-\lambda_1 t}}{\lambda_2 - \lambda_1} + e^{-\lambda_2 t} \cdot \text{const.}$$

$$N_2^{(0)} = 0 \Rightarrow N_2 = \lambda_1 \left(\frac{e^{-\lambda_1 t}}{\lambda_2 - \lambda_1} + \frac{e^{-\lambda_2 t}}{\lambda_1 - \lambda_2} \right) N_1^{(0)} = C_1^{(2)} e^{-\lambda_1 t} + C_2^{(2)} e^{-\lambda_2 t}$$

$$= \sum_{m=1}^2 C_m^{(2)} e^{-\lambda_m t} \quad \dots(3)$$

$$\frac{dN_3}{dt} + \lambda_3 N_3 = \lambda_2 N_2 \Rightarrow \frac{d(N_3 e^{\lambda_3 t})}{dt} = \lambda_2 \lambda_1 \left(\frac{e^{(\lambda_3 - \lambda_1)t}}{\lambda_2 - \lambda_1} + \frac{e^{(\lambda_3 - \lambda_2)t}}{\lambda_1 - \lambda_2} \right) N_1^{(0)}$$

Now we systematize the solution keeping in mind the method of partial fractions.

$$N_3 e^{\lambda_3 t} = \frac{\lambda_2 \lambda_1}{\lambda_1 - \lambda_2} \left(-\frac{e^{(\lambda_3 - \lambda_1)t}}{\lambda_3 - \lambda_1} + \frac{e^{(\lambda_3 - \lambda_2)t}}{\lambda_3 - \lambda_2} \right) N_1^{(0)} + \text{const.}$$

$$N_3^{(0)} = 0 \Rightarrow \text{const} = \frac{\lambda_2 \lambda_1}{\lambda_1 - \lambda_2} \left(\frac{1}{\lambda_3 - \lambda_1} - \frac{1}{\lambda_3 - \lambda_2} \right) N_1^{(0)}$$

$$= \lambda_1 \lambda_2 N_1^{(0)} \left\{ \left[\frac{1}{\lambda_2 - \lambda_3} \right]_{\lambda_3=\lambda_1} \frac{1}{(\lambda_1 - \lambda_3)} + \left[\frac{1}{\lambda_1 - \lambda_3} \right]_{\lambda_3=\lambda_2} \frac{1}{(\lambda_2 - \lambda_3)} \right\}$$

$$= \lambda_1 \lambda_2 \cdot \text{partial fractions of } \frac{1}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)} \cdot N_1^{(0)} = \frac{\lambda_1 \lambda_2 N_1^{(0)}}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)}$$

$$N_3 = \lambda_1 \lambda_2 \left(\frac{e^{-\lambda_1 t}}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)} + \frac{e^{-\lambda_2 t}}{(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_2)} + \frac{e^{-\lambda_3 t}}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)} \right) N_1^{(0)}$$

$$= C_1^{(3)} e^{-\lambda_1 t} + C_2^{(3)} e^{-\lambda_2 t} + C_3^{(3)} e^{-\lambda_3 t} = \sum_{m=1}^3 C_m^{(3)} e^{-\lambda_m t} \quad \dots(4)$$

$$\begin{aligned}
\frac{dN_4}{dt} + \lambda_4 N_4 &= \lambda_1 \lambda_2 \lambda_3 \left(\frac{e^{-\lambda_1 t}}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)} + \frac{e^{-\lambda_2 t}}{(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_2)} + \frac{e^{-\lambda_3 t}}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)} \right) N_1^{(0)} \\
\Rightarrow \frac{d(N_4 e^{\lambda_4 t})}{dt} &= \lambda_1 \lambda_2 \lambda_3 \left(\frac{e^{(\lambda_4 - \lambda_1)t}}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)} + \frac{e^{(\lambda_4 - \lambda_2)t}}{(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_2)} + \frac{e^{(\lambda_4 - \lambda_3)t}}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)} \right) N_1^{(0)} \\
\Rightarrow N_4 e^{\lambda_4 t} &= \lambda_1 \lambda_2 \lambda_3 \left(\frac{e^{(\lambda_4 - \lambda_1)t}}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)(\lambda_4 - \lambda_1)} + \frac{e^{(\lambda_4 - \lambda_2)t}}{(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_2)(\lambda_4 - \lambda_2)} \right. \\
&\quad \left. + \frac{e^{(\lambda_4 - \lambda_3)t}}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)(\lambda_4 - \lambda_3)} \right) N_1^{(0)} + \text{const.} \\
\Rightarrow N_4 &= \lambda_1 \lambda_2 \lambda_3 \left(\frac{e^{(-\lambda_1)t}}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)(\lambda_4 - \lambda_1)} + \frac{e^{(-\lambda_2)t}}{(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_2)(\lambda_4 - \lambda_2)} \right. \\
&\quad \left. + \frac{e^{(-\lambda_3)t}}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)(\lambda_4 - \lambda_3)} \right) N_1^{(0)} + \text{const.} e^{-\lambda_4 t} \\
N_4^{(0)} = 0 \Rightarrow \text{const} &= \lambda_1 \lambda_2 \lambda_3 \left(\frac{1}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)(\lambda_1 - \lambda_4)} + \frac{1}{(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_4)} \right. \\
&\quad \left. + \frac{1}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)(\lambda_3 - \lambda_4)} \right) N_1^{(0)} \\
&= \lambda_1 \lambda_2 \lambda_3 \left\{ \left[\frac{1}{(\lambda_2 - \lambda_4)(\lambda_3 - \lambda_4)} \right]_{\lambda_4=\lambda_1} \frac{1}{(\lambda_1 - \lambda_4)} + \left[\frac{1}{(\lambda_1 - \lambda_4)(\lambda_3 - \lambda_4)} \right]_{\lambda_4=\lambda_2} \frac{1}{(\lambda_2 - \lambda_4)} \right. \\
&\quad \left. + \left[\frac{1}{(\lambda_1 - \lambda_4)(\lambda_2 - \lambda_4)} \right]_{\lambda_4=\lambda_3} \frac{1}{(\lambda_3 - \lambda_4)} \right\} N_1^{(0)} \\
&= \lambda_1 \lambda_2 \lambda_3 \cdot \text{partial fractions of } \frac{1}{(\lambda_1 - \lambda_4)(\lambda_2 - \lambda_4)(\lambda_3 - \lambda_4)} \cdot N_1^{(0)} \\
&= \lambda_1 \lambda_2 \lambda_3 \cdot \frac{1}{(\lambda_1 - \lambda_4)(\lambda_2 - \lambda_4)(\lambda_3 - \lambda_4)} \cdot N_1^{(0)} \\
\Rightarrow N_4 &= \lambda_1 \lambda_2 \lambda_3 \left(\frac{e^{-\lambda_1 t}}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)(\lambda_4 - \lambda_1)} + \frac{e^{-\lambda_2 t}}{(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_2)(\lambda_4 - \lambda_2)} \right. \\
&\quad \left. + \frac{e^{-\lambda_3 t}}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)(\lambda_4 - \lambda_3)} + \frac{e^{-\lambda_4 t}}{(\lambda_1 - \lambda_4)(\lambda_2 - \lambda_4)(\lambda_3 - \lambda_4)} \right) N_1^{(0)} \\
&= C_1^{(4)} e^{-\lambda_1 t} + C_2^{(4)} e^{-\lambda_2 t} + C_3^{(4)} e^{-\lambda_3 t} + C_4^{(4)} e^{-\lambda_4 t} = \sum_{m=1}^4 C_m^{(4)} e^{-\lambda_m t} \quad \dots(5)
\end{aligned}$$

$$\frac{dN_5}{dt} = \lambda_1 \lambda_2 \lambda_3 \lambda_4 \left(\frac{e^{-\lambda_1 t}}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)(\lambda_4 - \lambda_1)} + \frac{e^{-\lambda_2 t}}{(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_2)(\lambda_4 - \lambda_2)} \right. \\ \left. + \frac{e^{-\lambda_3 t}}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)(\lambda_4 - \lambda_3)} + \frac{e^{-\lambda_4 t}}{(\lambda_1 - \lambda_4)(\lambda_2 - \lambda_4)(\lambda_3 - \lambda_4)} \right) N_1^{(0)}$$

$$N_5^{(0)} = 0 \Rightarrow N_5 = \lambda_1 \lambda_2 \lambda_3 \lambda_4 \left(\frac{1}{\lambda_1} \frac{1 - e^{-\lambda_1 t}}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)(\lambda_4 - \lambda_1)} + \frac{1}{\lambda_2} \frac{1 - e^{-\lambda_2 t}}{(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_2)(\lambda_4 - \lambda_2)} \right. \\ \left. + \frac{1}{\lambda_3} \frac{1 - e^{-\lambda_3 t}}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)(\lambda_4 - \lambda_3)} + \frac{1}{\lambda_4} \frac{1 - e^{-\lambda_4 t}}{(\lambda_1 - \lambda_4)(\lambda_2 - \lambda_4)(\lambda_3 - \lambda_4)} \right) N_1^{(0)}$$

It is not cumbersome to solve the first and last equations of the system Eq.(1) . The intermediate equations require some labour. The mathematics becomes somewhat non tedious if we consider the symmetries. The generalization of any intermediate equation is straightforward and we have:

$$\frac{dN_k}{dt} = -\lambda_k N_k + \lambda_{k-1} N_{k-1} \quad \dots(6)$$

Here the first term corresponds to its decay and the second one is related to its formation. The only assumption is :

$$N_1^{(0)} \neq 0, \forall N_j^{(0)} = 0, \text{ for } j > 1. \quad \dots(7)$$

Which is equivalent to :

$$N_j^{(0)} = N_1^{(0)} \delta_{ji} \quad \dots(8)$$

Looking at the intermediate solutions, we can write

$$N_k = \sum_{m=1}^k C_m^{(k)} e^{-\lambda_m t} = \{\lambda_1 \lambda_2 \lambda_3 \dots \lambda_{k-1}\} \left(\frac{e^{-\lambda_1 t}}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1) \dots (\lambda_k - \lambda_1)} + \frac{e^{-\lambda_2 t}}{(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_2) \dots (\lambda_k - \lambda_2)} \right. \\ \left. + \frac{e^{-\lambda_3 t}}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3) \dots (\lambda_k - \lambda_3)} \dots + \frac{e^{-\lambda_k t}}{(\lambda_1 - \lambda_k)(\lambda_2 - \lambda_k) \dots (\lambda_{k-1} - \lambda_k)} \right) N_1^{(0)} \quad \dots(9)$$

$$N_k = \{\lambda_1 \lambda_2 \lambda_3 \dots \lambda_{k-1}\} (C_1 e^{-\lambda_1 t} + C_2 e^{-\lambda_2 t} + C_3 e^{-\lambda_3 t} \dots + C_k e^{-\lambda_k t}) N_1^{(0)} = N_1^{(0)} \prod_{m=1}^{k-1} \lambda_m \cdot \sum_{l=1}^k C_l \cdot e^{-\lambda_l t} \quad \dots(10)$$

The symmetry in mathematical structure of Eq. (9) is utilized in this paper to further concise the solution of Eq.(1).

We have to seek similarity with calculus of residue and we obtained the following relationship.

$$\begin{aligned}
 C_p(\lambda_p) &= \frac{1}{(\lambda_1 - \lambda_p)(\lambda_2 - \lambda_p) \dots (\lambda_{p-1} - \lambda_p)} = \frac{(-1)^{p-1}}{(\lambda_p - \lambda_1)(\lambda_p - \lambda_2) \dots (\lambda_p - \lambda_{p-1})} \\
 C_p(\lambda) &= \frac{(-1)^{p-1}}{(\lambda - \lambda_1)(\lambda - \lambda_2) \dots (\lambda - \lambda_{p-1})} = \lim_{\lambda \rightarrow \lambda_p} \frac{(-1)^{p-1}(\lambda - \lambda_p)}{(\lambda - \lambda_1)(\lambda - \lambda_2) \dots (\lambda - \lambda_p)} \\
 &= (-1)^{p-1} \lim_{\lambda \rightarrow \lambda_p} (\lambda - \lambda_p) \frac{1}{(\lambda - \lambda_1)(\lambda - \lambda_2) \dots (\lambda - \lambda_p)} \\
 &= (-1)^{p-1} \cdot \text{residue of } \frac{1}{(\lambda - \lambda_1)(\lambda - \lambda_2) \dots (\lambda - \lambda_p)} \text{ at } \lambda = \lambda_p \\
 &= (-1)^{p-1} \cdot \text{residue of } \prod_{n=1}^p (\lambda - \lambda_n)^{-1} \text{ at } \lambda = \lambda_p
 \end{aligned}$$

$$\begin{aligned}
 N_k &= \{\lambda_1 \lambda_2 \lambda_3 \dots \lambda_{k-1}\} (C_1 e^{-\lambda_1 t} + C_2 e^{-\lambda_2 t} + C_3 e^{-\lambda_3 t} \dots + C_k e^{-\lambda_k t}) N_1^0 \\
 &= N_1^0 \sum_{n=1}^k e^{-\lambda_n t} \left\{ \prod_{i=1}^{k-1} (-\lambda_i) \right\} \cdot \text{residue of } \prod_{n=1}^k (\lambda - \lambda_n)^{-1} \text{ at } \lambda = \lambda_k \\
 &= N_1^0 \sum_{n=1}^k \cdot \text{residue of } e^{-\lambda_n t} \left\{ \prod_{i=1}^{k-1} (-\lambda_i) \right\} \prod_{n=1}^k (\lambda - \lambda_n)^{-1} \text{ at } \lambda = \lambda_k \\
 &= \text{sum of residues of } N_1^0 e^{-\lambda_n t} \left\{ \prod_{i=1}^{k-1} (-\lambda_i) \right\} \prod_{n=1}^k (\lambda - \lambda_n)^{-1} \text{ up to } \lambda = \lambda_k
 \end{aligned}$$

We define the function $f(z)$ in the following manner :

$$f(z) = N_1^0 e^{-\lambda_n t} \left\{ \prod_{i=1}^{k-1} (-\lambda_i) \right\} \prod_{n=1}^k (z - \lambda_n)^{-1} \quad \dots(11)$$

$$N_k = \frac{1}{2\pi i} \oint_C f(z) dz \quad \dots(12)$$

This Eq. (12) is familiar Cauchy's integral formula [8],[9].

CONCLUSION

In this way we succeeded in formulating Bateman solution as Cauchy's integral formula. Mathematical singularities suggested peculiar instabilities in the physical system. The singular points in λ - domain, are the localities of radioactive decay of the physical system. Mathematical singularities result in instabilities as a measure of radioactive transformations. In time domain radioactivity is described by half life times. Time domain is a continuum and λ - domain is discrete.

Conflict of Interest: The author declares no conflict of interest in relation to this research.

Data Availability Statement: Not applicable.

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